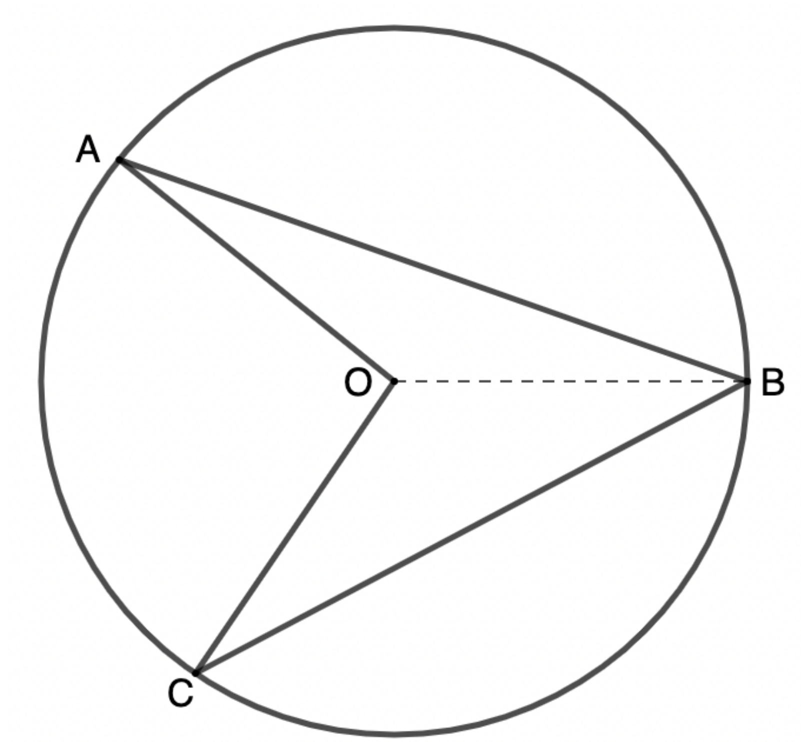
Geometry 2

Kirn Hans

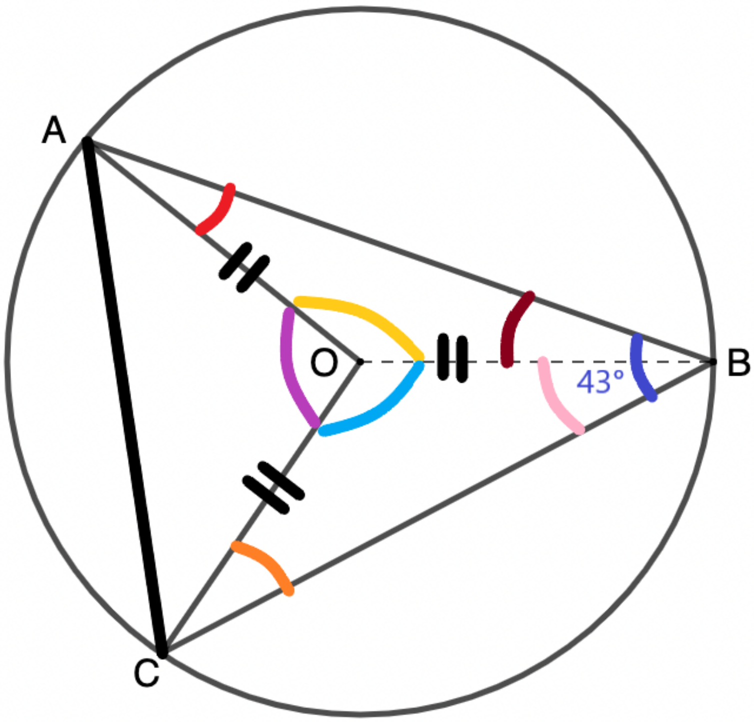
March 2021

Q. The diagram shows a circle with centre O and points A, B and C on its circumference.



The size of $\angle$ABC is 43 degrees. (diagram is not drawn to scale)
Find the size of $\angle$AOC, to the nearest degree.

A. Let's start by drawing a line between A and C, so we are dealing with triangles. I've also highlighted the key angles.



(Note: These colours have been chosen to account for red-green colourblindness, the most common type of colourblindness. Use of colour is meant to aid reading of angles.)

OB, OA and OC are all radii of the circle. That means they are equal in length, making △OBC and △OBA isosceles triangles.
As such,

$$\angle BCO = \angle OBC \tag{1}$$

And

$$\angle BAO = \angle OBA \tag{2}$$

Now, by the properties of triangles, we know that

$$\angle BOC + \angle BCO + \angle OBC = 180°$$

Similarly, we know that

$$\angle BOA + \angle BAO + \angle OBA = 180°$$

Adding these equations together,

$$\angle BOC + \angle BCO + \angle OBC + \angle BOA + \angle BAO + \angle OBA = 180° + 180°$$

We can combine $\angle OBC$ and $\angle OBA$ to make $\angle ABC$:

$\angle BOC + \angle BCO + \boxed{\angle BOA} + \angle BAO + \angle ABC = 360°$

We know $\angle ABC$ is 43°:

$\angle BOC + \angle BCO + \boxed{\angle BOA} + \angle BAO + 43° = 360°$

Simplifying:

$$\angle BOC + \angle BCO + \boxed{\angle BOA} + \angle BAO = 317° \tag{3}$$

From the equalities (1) and (2) we found earlier, we can substitute $\angle BCO$ and $\angle BAO$ in (3) to get:

$\angle BOC + \boxed{\angle OBC} + \boxed{\angle BOA} + \angle OBA = 317°$

Again, we can combine $\boxed{\angle OBC}$ and $\angle OBA$ to make $\angle ABC$, and replace it with its value of 43°:

$\angle BOC + \boxed{\angle BOA} + 43° = 317°$

Simplifying:

$$\angle BOC + \boxed{\angle BOA} = 274° \tag{4}$$

Now, all the angles around the point O must sum up to 360°, since they are angles around a single point. (This is because we define degrees by saying that a circle has 360°.)

That means:

$$\boxed{\angle BOA} + \angle AOC + \angle BOC = 360° \tag{5}$$

Using (4) in (5),

$274° + \angle AOC = 360°$

Subtracing 274° from both sides,

$\angle AOC = 86°$

And so, our answer is $\underline{86°}$.

Measurement 1

Kirn Hans

March 2021

Q. A rectangular sheet of metal has dimensions 66 cm x 72 cm.

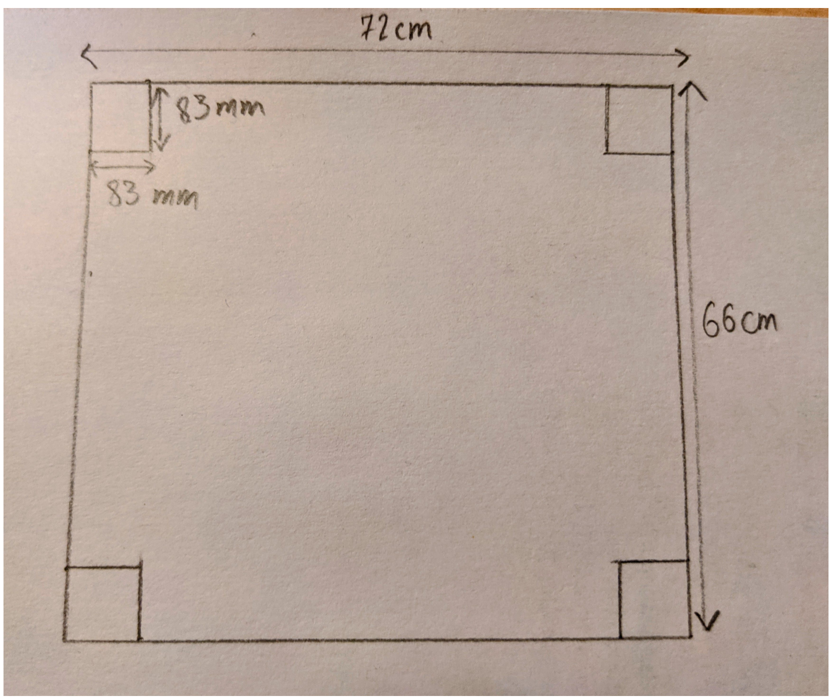
Squares of side length 83 mm are removed from each corner of the metal sheet.

The result is a 12-sided net of a rectangular box with no lid.

The box is formed by folding up the sides and soldering the joins together.

Find the amount of liquid that can be contained within this rectangular box, to the nearest litre.

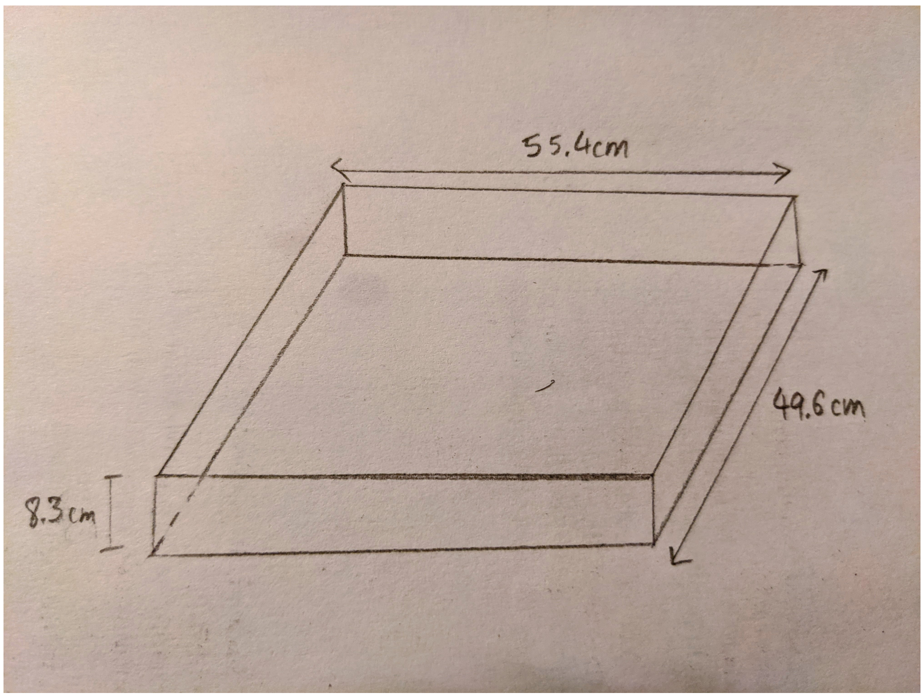
A. Let's determine the dimensions of the box.



For height, we look at the sides that fold up. The squares had side length 83 mm so the rectangular flap created on the side of the sheet has width of 83 mm, which becomes the height of the box. So, the height is 83 mm, or 8.3 cm.

For length, we look at the dimensions of the sheet, with the side flaps removed. Each flap has width 8.3 cm and there is one on each side of the rectangle. This means the flap width is removed twice from both length and breadth. So, length is 66 cm - (2×8.3 cm) = 49.6 cm

For breadth, we proceed the same way as we did with length, except with 72 cm instead of 66 cm. The breadth is 72 - (2×8.3 cm) = 55.4 cm



Now, to get the volume of the box, we multiply all three dimensions. Volume = 8.3 cm $\times$ 49.6 cm $\times$ 55.4 cm = 22,807.072 cm^3

We need the volume in litres. We know 1,000L is $1m^3$. We also know that 1 m = 100 cm, so, cubing both sides, 1 m^3 = 1,000,000 cm^3. Therefore, 1,000,000 cm^3 = 1,000L, or 1,000cm^3 = 1L

So we need to divide our volume 22,807.072 cm^3 by 1,000 to convert it to litres. Therefore, we have 22.807072L or, rounding to the nearest litre, 23L.

Our final answer is <u>23L</u>.

Measurement 2

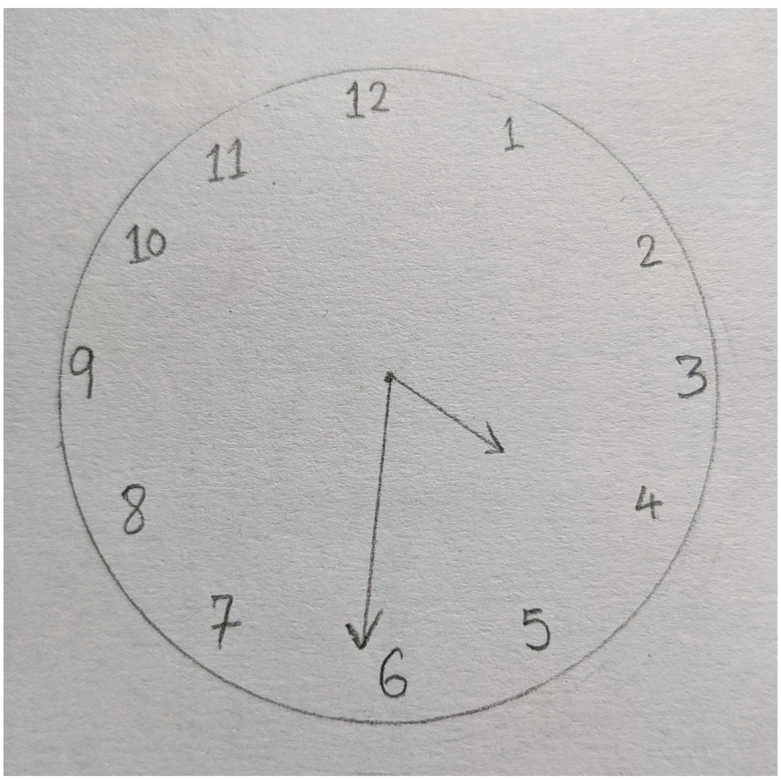
Kirn Hans

March 2021

Q. An analog clock is showing the time 4:32.

Find the size of the smaller angle between the hour hand and the minute hand, to the nearest degree.

A. Let's start by calculating the angles relative to the line between the centre of the clock and 12.



The angle of the minute hand at 32 is relatively straight-forward. There are 360 degrees in a circle. Our clock face is a circle representing 60 minutes. From this we get that each minute, the minute hand moves by $\frac{360°}{60} = 6°$.

So, when it's 4:32, the hand has moved $32 \times 6° = 192°$

Now for the hour hand. We might be tempted to say, there are 12 hours, so let's divide 360° by 12 and then multiply by 4 to get its position and we're done. This is certainly part of what we have to do. However, we must also remember that the hour hand does not abrupt swing to 5 as 4:59 changes to 5:00. It moves gradually throughout the hour.
How much has the hour hand moved beyond 4 by 4:32? Well, in the 60 minutes between 4:00 and 5:00, it will move the entire $\frac{360°}{12} = 30°$ between the 4 and 5 symbols on the edge of the clock. So the angle it's moved at 4:32 is some fraction of that.
What fraction? We know that 60 minutes is the whole hour, so our fraction is $\frac{32}{60}$.
Now with these two pieces of information, we are ready to calculate the position of the hour hand.
The position relative to 12 is

$$\begin{aligned} &= \frac{360°}{12} \times (4 + \frac{32}{60}) \\ &= 30° \times 4 + 30° \times \frac{32}{60} \\ &= 120° + \frac{32°}{2} \\ &= 136° \end{aligned}$$

Note that on the second line of this calculation, we used the distributive property to break up $(4 + \frac{32}{60})$ to make the calculation easier.

Now we know the angle of the minute hand relative to 12 and the angle of the hour hand relative to 12. To find the angle between the hands, we only need to find the difference of their angles relative to the same point, i.e. 12 in this case.
Thus, the angle between the hands $= 192° - 136° = 56°$
So our answer is $\underline{56°}$.

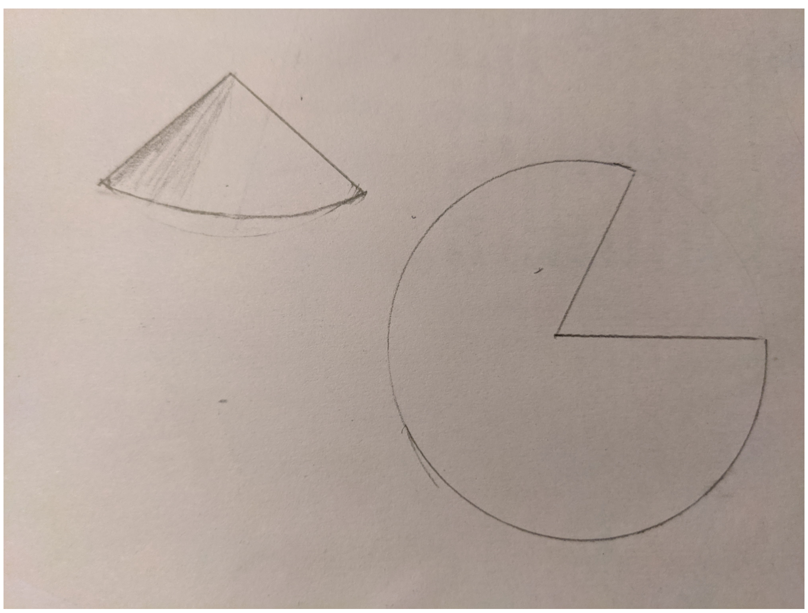
Measurement 3

Kirn Hans

March 2021

Q. A cone has radius 17 cm and slant height 22 cm.
The curved surface is flattened to form the sector of a circle.
Find the angle of the sector, to the nearest degree.

A. Flattening the curved surface means that the area of the sector is the surface area of the cone, without the base. We are looking at the net of the curved surface.



We know the surface area of a cone with radius r and slant height l is $\pi r(l+r)$.
πr^2 is the area of the circular base.
Therefore, the area of the curved surface is πrl. We can calculate its numeric value in a minute, but let's finish reasoning through this problem first and see if we can avoid calculations. (Remember, calculations are an opportunity to make a mistake and a good mathematician is a lazy mathematician!)

Now, because the cone was flattened to form the sector, the sector formed has the slant as its radius. If it were a whole circle and not a sector, we know the area would be πl^2. The angle of the sector determines its area relative to the area of a whole circle with the same radius. That is, if θ is the angle of the sector, the area of the sector

$$= \frac{\theta}{360^\circ} \pi l^2$$

For example, think of the area of a semicircle. Its angle is 180° and its area is $\frac{1}{2}$ of a whole circle.

So, to find θ, we need to divide the sector's area by the area of a circle with the same radius, then multiply by 360°:

$$\begin{aligned}
\theta &= \frac{\pi r l}{\pi l^2} \times 360^\circ \\
&= \frac{r}{l} \times 360^\circ \\
&= \frac{17}{22} \times 360^\circ \\
&= 278.18^\circ
\end{aligned}$$

So, our answer is $\underline{278^\circ}$.

Number 3

Kirn Hans

March 2021

Q. Joe gives 1/2 of his apples, plus 1/2 of an apple, to Jerry.
Joe then gives 1/3 of the remaining apples, plus 1/3 of an apple, to Julie.
Finally, Joe gives 1/4 of the remaining apples, plus 1/4 of an apple, to Jess.
In all of these transactions, none of the apples were cut.
What is the smallest number of apples that Joe could have started with?

A. Let the starting number of apples be x.

For none of the apples to have been cut, each quantity given has to be a whole number.

Let's calculate the quantities of apples that Joe gave his different friends with respect to x.

Amount given to Jerry

$$\begin{aligned} &= \frac{x}{2} + \frac{1}{2} \\ &= \frac{x+1}{2} \end{aligned}$$

To calculate the amount given to Julie, we have to first calculate how many apples were left after Jerry received his cut.
Remaining apples after Jerry

$$\begin{aligned} &= x - (\frac{x}{2} + \frac{1}{2}) \\ &= \frac{2x - x - 1}{2} \\ &= \frac{x-1}{2} \end{aligned}$$

We know that the amount given to Julie was $\frac{1}{2}$ of what was left and then $\frac{1}{2}$ an apple.

So, amount given to Julie

$$\begin{aligned}
&= \frac{1}{3} \times \frac{x-1}{2} + \frac{1}{3} \\
&= \frac{x-1}{6} + \frac{1}{3} \\
&= \frac{x-1+2}{6} \\
&= \frac{x+1}{6}
\end{aligned}$$

To calculate the amount given to Jess, again, we have to first calculate how many apples were left after Jerry and Julie received their cut.
So we subtract the amount Julie received from the amount of apples remaining after Jerry.
Remaining apples after Julie

$$\begin{aligned}
&= \frac{x-1}{2} - \frac{x+1}{6} \\
&= \frac{3(x-1)-(x+1)}{6} \\
&= \frac{3x-3-x-1}{6} \\
&= \frac{2x-4}{6} \\
&= \frac{x-2}{3}
\end{aligned}$$

$\frac{1}{4}$ of this and then $\frac{1}{4}$ of an apple was given to Jess, per the question.
So, amount given to Jess

$$\begin{aligned}
&= \frac{1}{4} \times \frac{x-2}{3} + \frac{1}{4} \\
&= \frac{x-2}{12} + \frac{1}{4} \\
&= \frac{x-2+3}{12} \\
&= \frac{x+1}{12}
\end{aligned}$$

We know all these amounts are whole numbers. We can use this to make deductions about divisibility of x.

So, looking at the amount given to each friend, we can see $\frac{x+1}{2}$, $\frac{x+1}{6}$ and $\frac{x+1}{12}$ are all whole numbers.

From this, we know 2, 6 and 12 all divide $x+1$, or to simplify, 12 divides $x+1$, since 2 and 6 are both factors of 12.

The smallest value of x for which 12 divides $x+1$ is 11, since 12 can divide itself but nothing smaller.

So, our answer is $\underline{\underline{11}}$.

Mathematics Teaching Methods 1
Assessment 1

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Patterns and Algebra 1

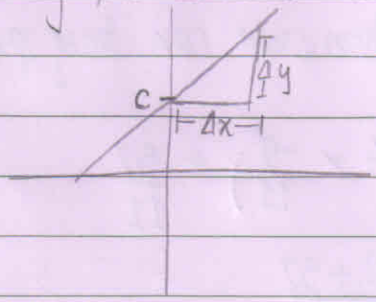
Q Find correct to two decimal places, the x-intercept of a straight line graph that passes through the points $(7,-1)$ and $(-4,8)$.

A. The formula for a straight line is given by

$$y = mx + c$$

where m is the slope, which indicates the angle of the line relative to x-axis
c is the y-intercept, or where the line crosses the y-axis
and y and x are variables, to be substituted by values on the line

We see this formula by visualising the line starting from its y-intercept and seeing the direction it goes.



The line crosses the y-axis at c so we get the point $(0,c)$ on the line. Now, to get the slope, we see how high the line goes (Δy) for how long it goes horizontally (Δx) in the same interval, i.e. for a second point on the line.

This ratio $\frac{\Delta y}{\Delta x}$ is known as the slope m.

Now that we have our formula, let us understand what the question is asking us. We have to find the x-intercept or the value of x at the point where $y = 0$.

From our two points, we can obtain the equation for this line.

From point $(7,-1)$, we know $-1 = 7m + c$ from the above formula
From point $(-4,8)$, $8 = -4m + c$

(Be careful at this point, it's easy to mix the order of x and y)

Now, we can solve for m and c as simultaneous equations.

To remove c, we can subtract the first equation from the second, because c's coefficient is the same in both equations.

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So: $8-(-1) = -4m + c - 7m - c$

$9 = -11m$

$-9/11 = m$

Putting this value in the first equation, we get:

$$-1 = \left(7 \times \frac{-9}{11}\right) + c$$

$$-1 + \frac{63}{11} = c$$

Swapping sides for clarity: $c = \frac{63-11}{11} = \frac{52}{11}$

To check our calculation, we can try putting these values in the second equation:

$$8 = \left(-4 \times \frac{-9}{11}\right) + \frac{52}{11}$$

$$= \frac{36+52}{11}$$

$$= \frac{88}{11}$$

$$= 8$$

The equation is true, so we have confirmed our values are right. Now, we can use these values in the equation $y = mx + c$ to get the x-intercept. We know $y = 0$ at this point so:

$$0 = \frac{-9}{11}x + \frac{52}{11}$$

Subtracting 52/11 from both sides:

$$-\frac{52}{11} = \frac{-9}{11}x$$

Dividing both sides by $-9/11$

$\frac{52}{9} = x$ at the x-intercept.

Calculating $\frac{52}{9}$ we get $5.7\overline{77}$ or, after rounding to the second decimal place: $\boxed{5.78}$

Patterns and Algebra 2

Kirn Hans

March 2021

Q. Find the coefficient of x^2 when $(5x+2)(2x-7)(3-9x)$ is fully expanded.

A. There are a few ways to do this. Of course, we can expand the entire expression. However, there is a shortcut.

We can see there are 3 terms containing x in the entire expression. We can only take one term from each group (and must take exactly one term per group). Each group is a unit that is being multiplied with another unit - no inter-multiplication within the unit. (This is a common mistake.)

However, we can divide this united group using the distributive principle: if multiplying the sum of two terms with a third is equivalent to summing the multiple of each term with the term.

To put it in notation:

$$(a+b) \times c = ac + bc$$

Now, with that in mind, let's see how which terms we want to look at multiplying together. If we multiply all 3 terms containing x, we get a term containing x^3.

If we multiply one term containing x with the constants of the other two groups, we get a term containing x.

If we multiply the constants together, we get a term without x all together.

Therefore, the only case we need to focus on is when we multiply two terms with x in them with constant of the third group.

There are 3 different pairs of terms containing x:

1. $5x$ and $2x$ (Taking the x terms from the first and second groups)
   When we take these terms from the first two groups, that leaves us with the constant of the third group: 3
   Therefore, we multiply all these together:
   $$5x \times 2x \times 3 = 30x^2$$
2. $2x$ and $-9x$ (Taking the x terms from the second and third groups)
   (A common mistake here is to forget the minus sign in front of the 9.)

Now we'll take the constant from the first group: 2

$$2x \times -9x \times 2 = -36x^2$$

(Again, careful to not drop the minus sign! I've made this error as a student; learn from my mistakes.)

3. $5x$ and $-9x$ (Taking the x terms from the first and third groups)
   Now we multiply with the constant of the second group: -7
   To be careful about my multiplication, I am going to do it two items at a time, since these operands are slightly bigger than the previous cases.

   $$5x \times -9x \times -7 = 45x^2 \times 7 = (280 + 35)x^2 = 315x^2$$

   Note how the minus signs of -9 and -7 cancel out.

Now we have all the x^2 terms in the expanded expression. How would we handle these terms if we were expanding as normal? Well, we would add them, so that's what we'll do with these three terms.

$$30x^2 + (-36x^2) + 315x^2 = 315x^2 - 6x^2 = 309x^2$$

And that is our final answer. The coefficient of x^2 in the expanded expression is:

$$\underline{309}$$

Patterns and Algebra 4

Kirn Hans

March 2021

Q. The parabola $y = 4x^2 + nx + 1$ crosses the x axis at two different points. What is the smallest possible positive integer value of n?

A. For the parabola to cross the x axis twice, there must be some portion of the parabola below the x axis, as opposed to a single point on the x axis or all points above the x axis.

For fun, imagine it like dipping a toe into water. You can submerge it, let it skim the surface or keep your foot far away from the water, like my cat. When you submerge your toe, the line between the water and the air touches both the front and back of the toe. In the same way, the x axis touches the parabola at two points.

So we know that for some values of x,

$$4x^2 + nx + 1 < 0$$

Rearranging the inequality,

$$4x^2 + 1 < -nx$$

This looks like it might be difficult to isolate n. Instead, let's consider what values of x might be useful to determine the value of n.

We know from the question that n is positive. Therefore, for a positive value of x, all terms of $4x^2 + nx + 1$ will be positive and so the whole expression will be positive.

If we put $x = 0$ in $y = 4x^2 + nx + 1$, we get $y = 1$, which is still positive.

So now we will start looking at negative values of x.

First, let's look at the term $4x^2$. Because it's a square with a positive coefficient, this term will always be positive, no matter the value of x.

A quick reminder: all squares of integers are positive because multiplying two negative numbers cancels out their negativity.

Also, the graph of x^2 grows at a much faster rate than the graph of x. The graphs intersect at $x = 1$ (and at $x = 0$ but we already determined 0 is not a helpful value of x).

So, knowing that we need a negative value of x and that x^2 leaves x behind after $x = 1$, we can put those two facts together and conclude that $x = -1$ might be a good starting point. We could look at values between 0 and -1, but -1 has the greatest absolute value of this range. $4x^2$ would be smaller for values of x in (-1,0), but nx would also be smaller. Therefore, let's proceed with $x = -1$.

If we put $x = -1$ into our expression, we get

$$4(-1)^2 + n(-1) + 1 = 5 - n$$

That looks promising. If $y = 0$ at $x = -1$, $n = 5$. Now here's a question: do we need $y < 0$ or $y = 0$ at $x = -1$?

The answer is, it depends on the expression y is equal to. If y has only one root, the parabola will not cross the x axis twice. It will only touch it once, at the root.

Let's look at the case where $n = 5$ and factor $4x^2 + 5x + 1$. If it has only one root, we'll know $n = 5$ is not the value we're looking for.

To get the roots of the expression, we can use the quadratic formula. The quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the equation $ax^2 + bx + c = 0$.
(If you're curious where this formula comes from, we get this formula by adjusting both sides to wrangle the equation into the form $(2ax + b)^2 = b^2 - 4ac$ so that we can isolate x.)

So, inserting our values of $a = 4$, $b = 5$, and $c = 1$:

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4 \times 4 \times 1}}{2 \times 4} \\ &= \frac{-5 \pm \sqrt{25 - 16}}{8} \\ &= \frac{-5 \pm \sqrt{9}}{8} \\ &= \frac{-5 \pm 3}{8} \end{aligned}$$

Simplifying, we get that $x = -1$ or $x = -\frac{1}{4}$

Thus, $4x^2 + 5x + 1$ has two roots. This answers our question of whether y has only one root if $n = 5$. It does not, so we are in the clear.
So, $\underline{5}$ is our answer.

Statistics and Probability 3

Kirn Hans

March 2021

Q. In a group of 61 students, 22 students wear watches and 15 students wear glasses.

6 students wear both a watch and glasses.

How many students do not wear either a watch or glasses?

A. We can begin by drawing a Venn Diagram to represent the information we have. This helps us get an big-picture view.